1 Introduction

This document investigates the theory behind the 'Tandem Bridge'. The bridge consists of only two components yet it is very complex and tedious to analyse.

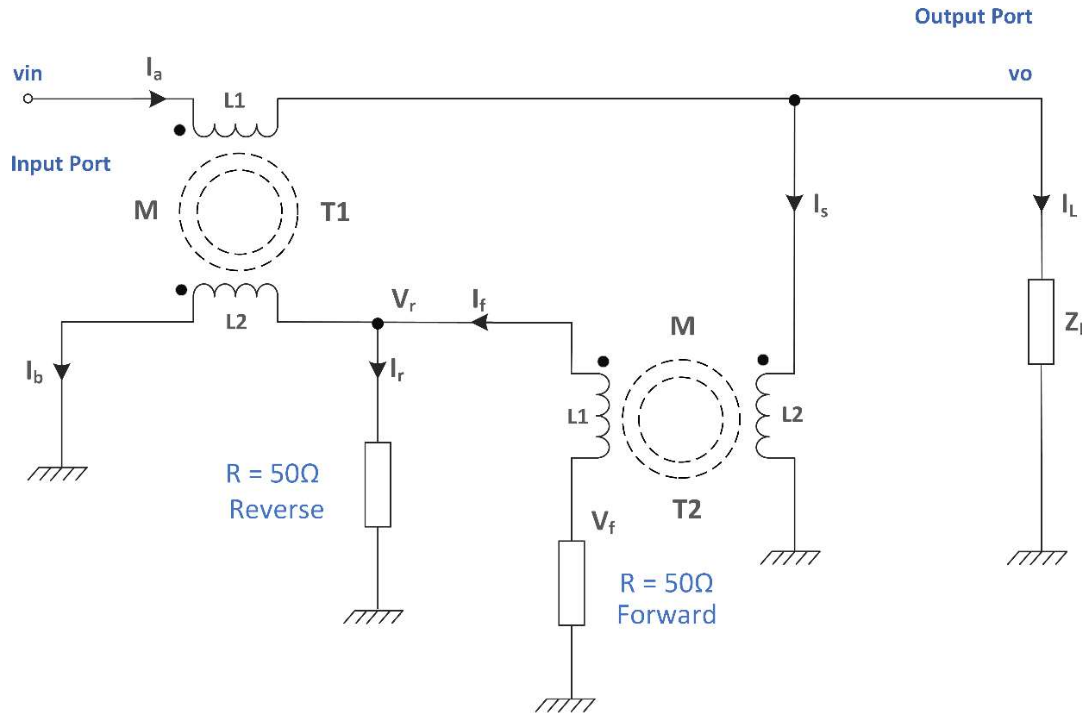


Figure 1 - The Basic Tandem Bridge

The tandem bridge is a compromise. The level of return loss depends upon on the winding inductance and the mutual coupling between them. At no time does the reverse voltage 'Vr' ever fall to zero even when the load is exactly 50Ω. This circuit does not balance as such. Observation of the circuit suggests that this bridge must also introduce an impedance change in the transmitter path. However, the tandem bridge is an excellent performer and deserves full analysis.

2 Key Equations

These four expressions define the key performance parameters of a Tandem bridge:

Reverse Voltage:

$$V_r = \frac{V_0\left(j\omega k\sqrt{L_1L_2}\left(1-\frac{R}{Z}\right)+\omega^2\frac{kL_1\sqrt{L_1L_2}}{Z}\left(1-k^2\right)-kR\sqrt{\frac{L_1}{L_2}}-j\omega kL_1\sqrt{\frac{L_1}{L_2}}\right)}{R+j2\omega L_2+j\omega L_1\left(1-2k^2\right)+\frac{\omega^2L_1L_2}{R}(k^2-1)}$$ (Volts) Equation 13

Forward Voltage:

$$V_f = \frac{V_r - kV_o\sqrt{\frac{L_1}{L_2}}}{1 + j\omega\frac{L_1}{R}\left(1-k^2\right)}$$

(Volt) Equation 12

Input Impedance:

$$Z_{in} = \frac{V_0 + j\omega\, k\sqrt{L_1 L_2}\left(\frac{V_f + V_r}{R}\right)}{\frac{V_o}{j\omega\, L_2} - \frac{k\sqrt{L_1 L_2}V_f}{RL_2} + \frac{V_o}{Z}} + j\omega\, L_1$$

(Ohms) Equation 14

Insertion Loss:

$$I_L = 20\log\left(\left|\frac{V_0}{V_0 + j\omega L_1\left(\frac{V_o}{j\omega L_2} - \frac{k\sqrt{L_1L_2}V_f}{RL_2} + \frac{V_o}{Z}\right) + j\omega k\sqrt{L_1L_2}\left(\frac{V_f + V_r}{R}\right)}\right|\right)$$

(dB) Equation 15

Appendix A of this document provides a derivation of Equations 12, 13, 14 and 15 (if you are keen). I avoided the further simplification offered by replacing the inductor values with the expressions involving the turns ration N. The inductor values will change with frequency and are more fundamental than the turns ratio N. I also avoided expressing the complex terms in canonical form - it is just not worth the time and effort, particularly when the expressions will be machine calculated.

2.1 Minimum Reflected Signal

Equations 13 can be simplified by making assumptions. Assume that the coupling function k = 1, Vo=1 and the output impedance Z = R. Under these 'balanced' conditions equation 13 becomes:

$$V_r = \frac{-R\sqrt{\frac{L_1}{L_2}} - j\omega L_1\sqrt{\frac{L_1}{L_2}}}{R + j2\omega L_2 + j\omega L_1(-1)} = \frac{-(R + j\omega L_1)\sqrt{\frac{L_1}{L_2}}}{R + j\omega(2L_2 - L_1)}$$

As the frequency increases:

$j\omega L_1 >> R$ and $j\omega(2L_2 - L_1) >> R$

So

$R + j\omega L_1 \to j\omega L_1$ and $R + j\omega(2L_2 - L_1) \to j\omega(2L_2 - L_1)$ respectively.

Hence, the minimum level of V_r is determined by the inductance values alone. Since the bridge must be practical with realistic inductance values, the minimum reflected level of the Tandem bridge is finite - the bridge never balances. The following expression illustrates the limit.

$$V_r \cong \frac{-j\omega L_1 \sqrt{\frac{L_1}{L_2}}}{j\omega(2L_2 - L_1)}$$

Assuming $L_2 >> L_1$ (In fact it is N x N larger where N is the turns ratio)

$$V_r \cong -\frac{1}{2}\frac{L_1}{L_2}\sqrt{\frac{L_1}{L_2}}$$ Lower limit for the reflected signal

The negative sign indicates that the reflected signal is antiphase relative to the output signal. To be sure, I investigated an LTspice evaluation using 'transient' analysis and confirmed that both the forward and reverse signal appeared in antiphase compared to the output signal.

As an example, if a Tandem bridge employs cores with a turns ration N=11 and the inductance of a single turn on the ferrite core is L_1=2.3uH then L_2 = 278.3uH and so $\sqrt{\frac{L_1}{L_2}} = \frac{1}{11}$.

An estimate for the lower limit is $V_r \cong \left(\frac{1}{2.11^3}\right)$ or $20Log_{10}\left(\frac{1}{2.11^3}\right) = -68.5$ dB below the output signal.

2.2 Forward Signal at Minimum Reflected Value

The forward voltage expression simplifies if k=1 and Vo=1. Also, if the reflected signal is low compared with the output voltage i.e. at -68.5dB then:

$$V_f = \frac{V_r - kV_o\sqrt{\frac{L_1}{L_2}}}{1 + j\omega\frac{L_1}{R}\left(1-k^2\right)}$$ Equation 12 Again

$$V_f \cong -\sqrt{\frac{L_1}{L_2}} \text{ or } 20Log_{10}\left(\frac{1}{11}\right) = -20.83 \text{ dB}$$ below the output signal

Hence, for N=11, the very best return loss is -47.67 dB.

2.3 Verification of Equation 12 and Equation 13

Equation 12 and 13 requires verification. One way to do this is to test the results against validated circuit simulation software that calculates similar results using a different but equivalent technical approach. I employed LTspice which is very capable and free. Inductors in LTspice are defined using mutual inductance in the same way as they are defined in equations 12 and 13, enabling an exact comparison. LTspice, of course, calculates numerical values that are presented on a graph. These need extracting using the cursor and the computer status bar.

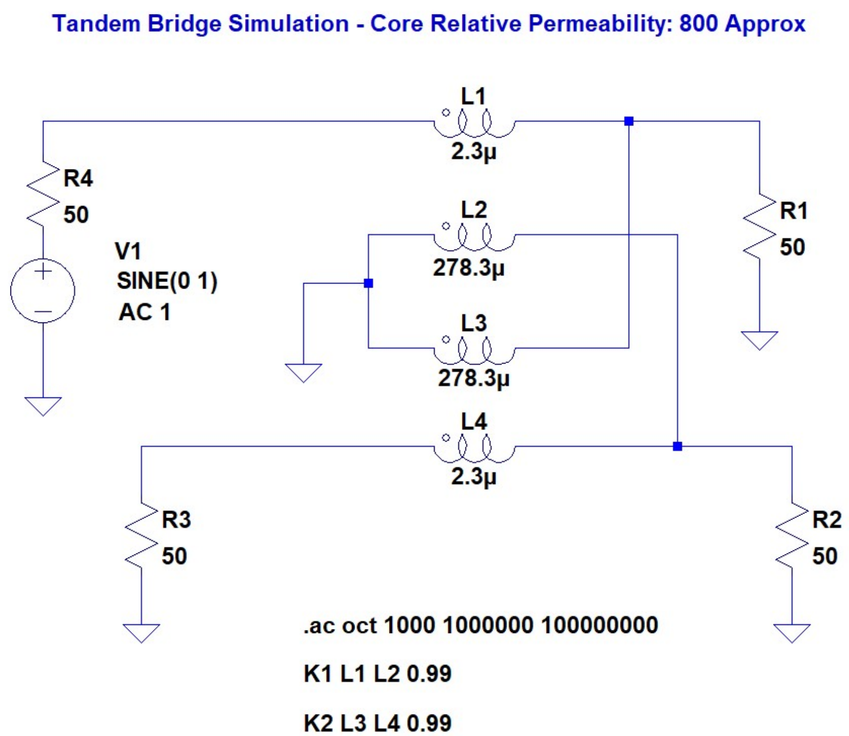


Figure 2 - LTspice Implementation of a Tandem Bridge

In the above LTspice example, if k=0.99, L1 =2.3µH, L2 = 278.3 µH (N=11) and f = 2MHz, then the forward, reverse and output signals calculated by LTspice are:

Output signal = - 6.057 dB (The output is effectively divided in half by R1 and R4)
Forward signal = - 26.936 dB
Reverse Signal = - 74.144 dB

Relative to the output signal (Add 6.057 dB to all three):

Output signal = 0 dB
Forward signal = - 20.879 dB
Reverse Signal = - 68.087 dB
Return Loss = - 47.208 dB

The calculation of equations 12 and 13 is best performed using capable mathematical software. The easiest and most universal way is probably Python or C. Any method will do as long as the complex numbers are properly implemented. It is best to avoid spreadsheets unless you have lots of time to spend and masochistic tendencies.

For this document, I used 'Mathcad' which is a bit specialised but it provides a WYSIWYG representation. This is easier to follow than lines of 'C' code.

As an example:

The Tandem Bridge

$\omega := 2\cdot\pi\cdot 2\cdot 10^{6} \quad k := 0.99 \quad R := 50 \quad Z := 50 \quad L1 := 2.3\cdot 10^{-6}$

$L2 := 278.3\cdot 10^{-6} \quad Vo := 1$

Calculate the forward and reverse voltages using Equation 13 and Equation 12

$$Vr := \frac{Vo\cdot\left[j\cdot\omega\cdot k\cdot\sqrt{L1\cdot L2}\cdot\left(1-\frac{R}{Z}\right)+\omega^{2}\cdot k\cdot L1\cdot\frac{\sqrt{L1\cdot L2}}{Z}\cdot\left(1-k^{2}\right)-k\cdot R\cdot\sqrt{\frac{L1}{L2}}-j\cdot\omega\cdot L1\cdot k\cdot\sqrt{\frac{L1}{L2}}\right]}{R+j\cdot 2\cdot\omega\cdot L2-j\cdot\omega\cdot L1\cdot\left(2\cdot k^{2}-1\right)+\frac{\omega^{2}\cdot L1\cdot L2\cdot\left(k^{2}-1\right)}{R}}$$

$$Vf := \frac{\left(Vr-k\cdot Vo\cdot\sqrt{\frac{L1}{L2}}\right)}{1+j\cdot\omega\cdot\frac{L1}{R}\cdot\left(1-k^{2}\right)}$$

$Vr = -3.7355861\times 10^{-4}+1.256984i\times 10^{-4}$

$Vf = -0.09+1.165i\times 10^{-3}$

$20\cdot\log(|Vf|) = -20.88$

$20\cdot\log(|Vr|) = -68.087$

Figure 3 - Mathcad Evaluation - Verification

The above Mathcad sheet uses the same parameters as the LTspice simulation to calculate the forward and reflected voltages. The results are correct to at least 2 decimal places. Similar calculations using different loads, different k values and different frequency always gave comparable results. I conclude therefore that equations 12 and 13 are *probably* correct. (*I say probably because these equations give steady state values and do not include transient terms that following immediate activation of the circuit*).

2.4 Tandem Bridge - Return Loss - Verification

Does the Tandem bridge provide an accurate rendition of return loss? For a transmission system with an arbitrary termination, the reflection coefficient at an impedance discontinuity is described as:

$$\rho = \frac{Z_L - Z_o}{Z_L + Z_o}$$

For the Tandem Bridge, the reflection coefficient is similarly described as $\rho = \frac{V_r}{V_f}$

In both cases, the Return Loss in dB is: $RL = 20.\log|\rho|$

Using k=0.99, L1 =2.3µH, L2 = 278.3 µH (N=11) and f = 2MHz as an example. The return loss for a range of Z (the load) can be plotted. The following Mathcad sheet does this for a range of Z real values from 0 to 200Ω. Complex imaginary values for Z can be added - in this example it is set to +j1. Figure 4 shows the results for a near resistive load.

The Tandem Bridge

$\omega := 2\cdot\pi\cdot 2\cdot 10^6$ $\quad k := 0.99$ $\quad R := 50$ $\quad n := 0..200$ $\quad X := 1$

$Z_n := n + j\cdot X$

$L1 := 2.3\cdot 10^{-6}$ $\quad L2 := 278.3\cdot 10^{-6}$ $\quad Vo := 1$

Calculate the forward and reverse voltages using Equation 13 and Equation 12

$$Vr_n := \frac{Vo\cdot\left[j\cdot\omega\cdot k\cdot\sqrt{L1\cdot L2}\cdot\left(1-\frac{R}{Z_n}\right)+\omega^2\cdot k\cdot L1\cdot\frac{\sqrt{L1\cdot L2}}{Z_n}\cdot\left(1-k^2\right)-k\cdot R\cdot\sqrt{\frac{L1}{L2}}-j\cdot\omega\cdot L1\cdot k\cdot\sqrt{\frac{L1}{L2}}\right]}{R+j\cdot 2\cdot\omega\cdot L2-j\cdot\omega\cdot L1\cdot\left(2\cdot k^2-1\right)+\frac{\omega^2\cdot L1\cdot L2\cdot\left(k^2-1\right)}{R}}$$

$$Vf_n := \frac{\left(Vr_n - k\cdot Vo\cdot\sqrt{\frac{L1}{L2}}\right)}{1+j\cdot\omega\cdot\frac{L1}{R}\cdot\left(1-k^2\right)}$$

$$RL_n := 20\cdot\log\left(\left|\frac{Vr_n}{Vf_n}\right|\right)$$

Theoretical return Loss

$$p_n := \frac{Z_n - R}{Z_n + R}$$

$$RLt_n := 20\cdot\log\left(\left|p_n\right|\right)$$

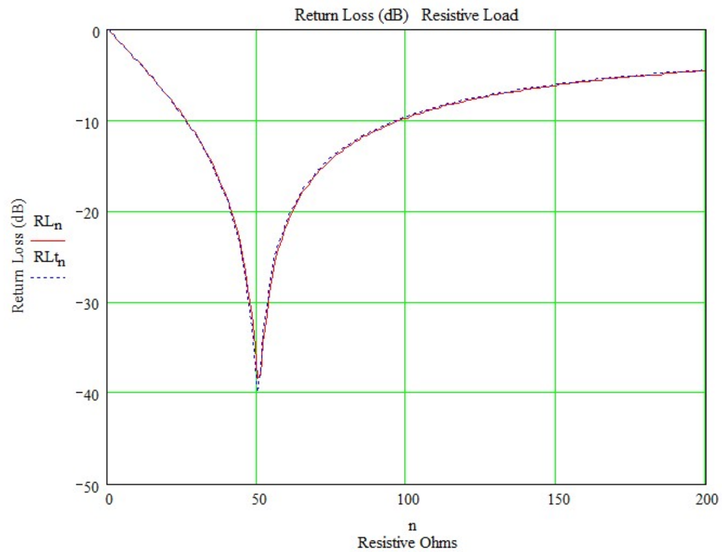


Figure 4 - Mathcad Worksheet - Reactance of +j1Ω (Essentially Resistive Load)

Figure 4 shows the theoretical return losses and the values returned by the Tandem Bridge. They appear very close. The reactive part of the load impedance 'Z' was set to + j1Ω for convenience to avoid divide by zero errors. However, the 1Ω reactive component reduced the return loss from -47dB at Z=50Ω to -40dB at Z = 50 +j1Ω! The bridge is very sensitive!

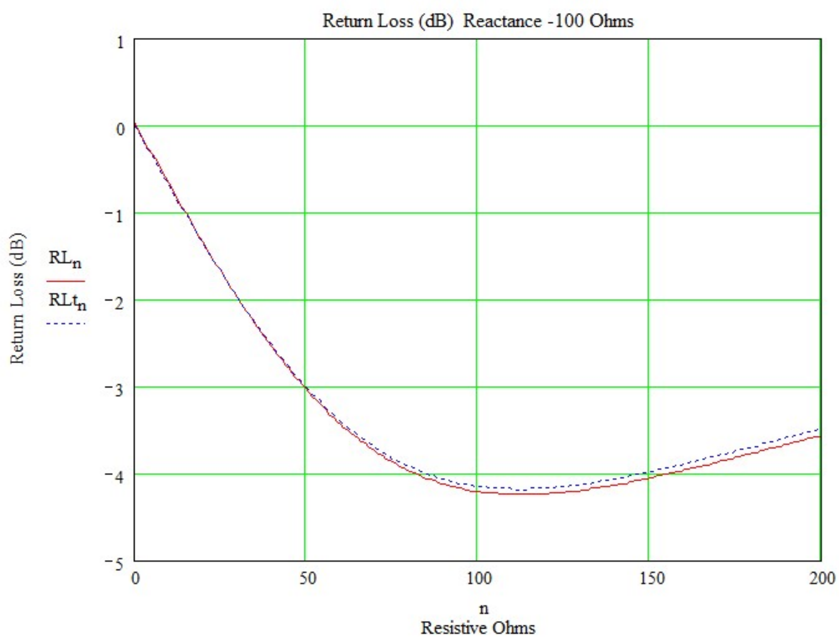


Figure 5 - Mathcad Worksheet - Reactance of +1Ω (Essentially Resistive load)

Figure 5 shows the theoretical and the Tandem bridge return loss for Z= (0 to 200) - j100Ω. Again, the two values are very close. I conclude that the Tandem bridge provides a good representation of the forward and reflected signals. This conclusion is intuitive - the signals at the forward and reflected port are just a transformer-reduced constructs of the primary circuit.

2.5 Tandem Bridge - Input Impedance - Verification

The input impedance of the bridge can be calculated using equation 14:

$$Z_{in} = \frac{V_0 + j\omega\, k\sqrt{L_1 L_2}\left(\dfrac{V_f + V_r}{R}\right)}{\dfrac{V_o}{j\omega\, L_2} - \dfrac{k\sqrt{L_1 L_2}\, V_f}{R L_2} + \dfrac{V_o}{Z}} + j\omega\, L_1$$

(Ohms) Equation 14

This can be verified against an LTspice simulation. For example, using the same parameters as before: k=0.99, L1 =2.3µH, L2 = 278.3 µH (N=11), Z=50Ω and f = 1 to 50 MHz, the Mathcad sheet calculates the input impedance and plots it onto a graph against frequency.

Input impedance results from the LTspice simulation were taken manually at 2, 10, 20, 30, 40 and 50 MHz using the graph and status bar. These were plotted onto the same Mathcad graph and shown as blue circles.

The results were identical to three decimal places or more. At low frequencies, the input impedance magnitude of the bridge is less than 50Ω probably due to the shunting effect of L2 across the output. At higher frequency, the series impedance of L1 starts to dominate.

The input impedance is easily converted into VSWR. For the condition stated, such a bridge would provide an insertion VSWR of less than 1.2 at frequencies below 30MHz.

The data provided here is for verification only. The evaluation of a practical bridge can be made based on real inductors and real ‘k’ values.

The Tandem Bridge - Input Impedance

$k := 0.99 \qquad R := 50 \qquad n := 0..50000$

$f_n := 1 + n \cdot 0.001$

$\omega_n := 2 \cdot \pi \cdot (f_n) \cdot 10^6$

$Z := 50$

$L1 := 2.3 \cdot 10^{-6} \qquad L2 := 278.3 \cdot 10^{-6} \qquad Vo := 1$

Calculate the forward and reverse voltages using Equation 13 and Equation 12

$$Vr_n := \frac{Vo \cdot \left[j \cdot \omega_n \cdot k \cdot \sqrt{L1 \cdot L2} \cdot \left(1 - \frac{R}{Z}\right) + (\omega_n)^2 \cdot k \cdot L1 \cdot \frac{\sqrt{L1 \cdot L2}}{Z} \cdot \left(1 - k^2\right) - k \cdot R \cdot \sqrt{\frac{L1}{L2}} - j \cdot \omega_n \cdot L1 \cdot k \cdot \sqrt{\frac{L1}{L2}} \right]}{R + j \cdot 2 \cdot \omega_n \cdot L2 - j \cdot \omega_n \cdot L1 \cdot \left(2 \cdot k^2 - 1\right) + \frac{(\omega_n)^2 \cdot L1 \cdot L2 \cdot \left(k^2 - 1\right)}{R}}$$

$$Vf_n := \frac{\left(Vr_n - k \cdot Vo \cdot \sqrt{\frac{L1}{L2}}\right)}{1 + j \cdot \omega_n \cdot \frac{L1}{R} \cdot \left(1 - k^2\right)}$$

$$Zin_n := \frac{Vo + j \cdot \omega_n \cdot k \cdot \sqrt{L1 \cdot L2} \cdot \left(\frac{Vf_n + Vr_n}{R}\right)}{\frac{Vo}{j \cdot \omega_n \cdot L2} - \frac{k \cdot \sqrt{L1 \cdot L2} \cdot Vf_n}{R \cdot L2} + \frac{Vo}{Z}} + j \cdot \omega_n \cdot L1$$

From LTspice Simulation

$LT_{0,0} := 2 \qquad LT_{0,1} := 49.605$

$LT_{1,0} := 10 \qquad LT_{1,1} := 49.692$

$LT_{2,0} := 20 \qquad LT_{2,1} := 49.947$

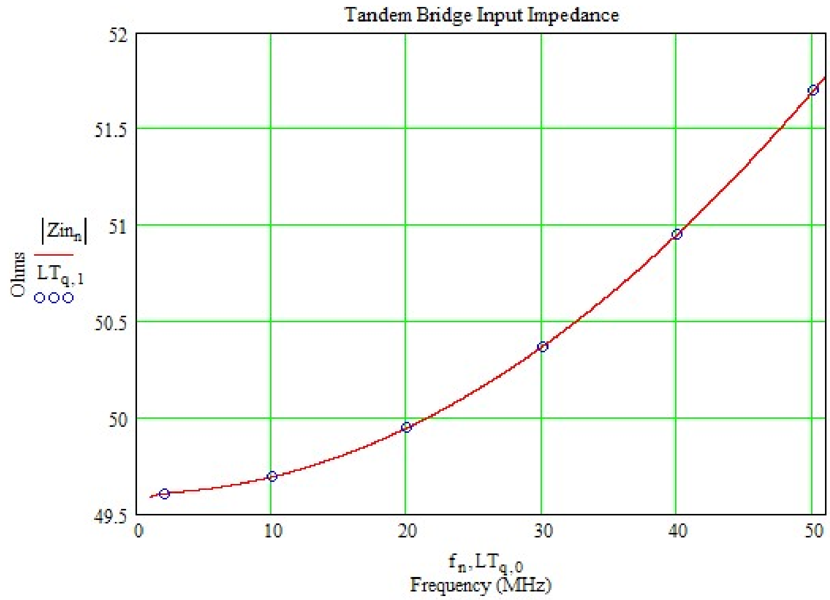
$LT_{3,0} := 30 \qquad LT_{3,1} := 50.369$

$LT_{4,0} := 40 \qquad LT_{4,1} := 50.954$

$LT_{5,0} := 50 \qquad LT_{5,1} := 51.696$

$q := 0..5$ q steps through the LTspice data

Figure 6 - Mathcad Calculation of Input Impedance



$$p_n := \frac{Zin_n - Z}{Zin_n + Z} \qquad s_n := \frac{1 + |p_n|}{1 - |p_n|}$$

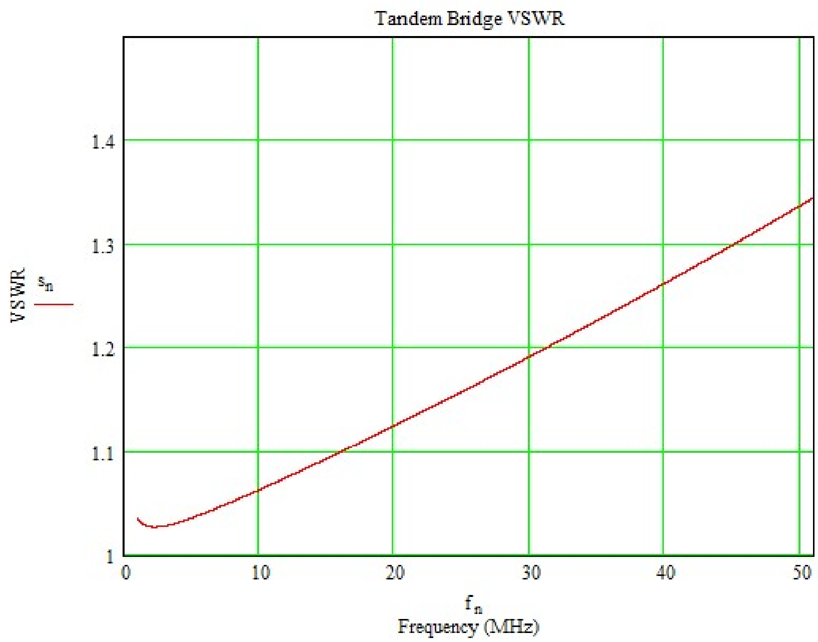


Figure 7 - Tandem Bridge - Input impedance and Input VSWR

2.6 Tandem Bridge - Insertion Loss - Verification

The insertion loss of the bridge can be calculated using Equation 15. This can be verified against an LTspice simulation. For example, using the same parameters as before: k=0.99, L1 =2.3µH, L2 = 278.3 µH (N=11), Z=50Ω and f = 1 to 50 MHz, the Mathcad sheet calculates the insertion loss and plots it onto a graph against frequency.

$$I_L = 20\log\left(\left|\frac{V_0}{V_0 + j\omega L_1\left(\frac{V_o}{j\omega L_2} - \frac{k\sqrt{L_1L_2}V_f}{RL_2} + \frac{V_o}{Z}\right) + j\omega k\sqrt{L_1L_2}\left(\frac{V_f + V_r}{R}\right)}\right|\right) \quad \text{(dB)} \qquad \text{Equation 15}$$

Insertion Loss

From LTspice Simulation

$ILS_{0,0} := 2$	$ILS_{0,1} := 0.999734$
$ILS_{1,0} := 10$	$ILS_{1,1} := 0.998093$
$ILS_{2,0} := 20$	$ILS_{2,1} := 0.993018$
$ILS_{3,0} := 30$	$ILS_{3,1} := 0.984730$
$ILS_{4,0} := 40$	$ILS_{4,1} := 0.973472$
$ILS_{5,0} := 50$	$ILS_{5,1} := 0.959555$
$q := 0..5$	q steps through the LTspice data

$$IL_n := 20\cdot\log\left[\left|\frac{Vo}{Vo + j\cdot\omega_n\cdot L1\cdot\left(\frac{Vo}{j\cdot\omega_n\cdot L2} - \frac{k\cdot\sqrt{L1\cdot L2}\cdot Vf_n}{R\cdot L2} + \frac{Vo}{Z}\right) + j\cdot\omega_n\cdot k\cdot\sqrt{L1\cdot L2}\cdot\left(\frac{Vf_n + Vr_n}{R}\right)}\right|\right]$$

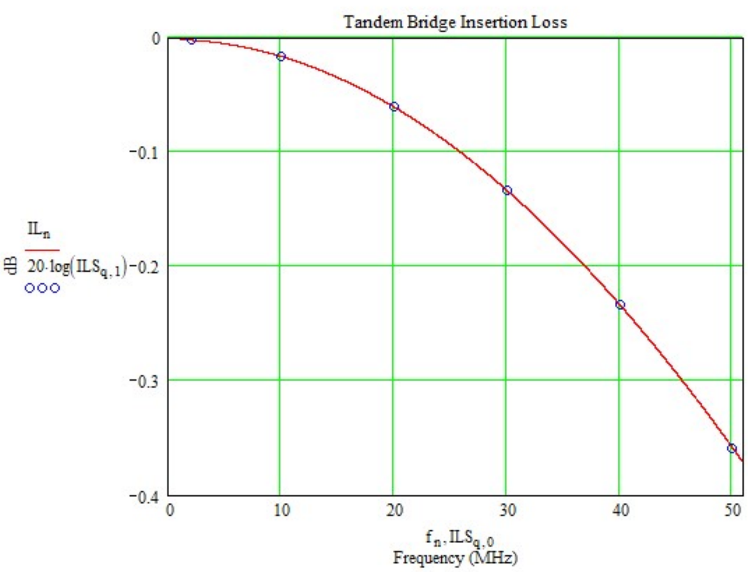


Figure 8 - Equation 15 and LTspice Provides Identical Results

The results are essentially identical. I conclude the four equations 12, 13, 14 and 15 give an accurate mathematical description of a Tandem bridge.

3 Appendix A - Outline of a Proof for Equation 12, 13, 14 and 15 - Preliminaries

Mutual inductance is necessary to analyse coupled circuits. To be successful, it is essential to follow a set of rules and remain consistent in applying signs throughout. It is necessary to guess the initial current directions and then calculate associated voltages accepting that some guesses may be incorrect. If the analysis is consistent and applied robotically, the analysis works out correctly in the wash. The voltages induce into windings come in two forms:

(1) self-induced voltages (you could call these back-emfs if you have too); and

(2) mutually induced voltage that are generated by changing currents in the other winding.

The relative polarities of the voltages are described by the dot convention as shown in Figure 9. Of course, all voltages and currents are assumed RF sinusoids.

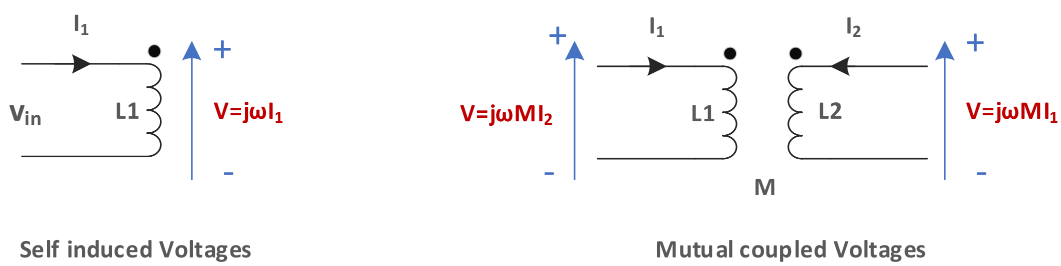


Figure 9 - Application of the Dot Convention

Transformer Dot Convention for mutual inductance is described in words as:

- If a current 'enters' into the dotted terminal of a coil, then the polarity of the induced voltage in the other coil is positive at its dotted terminal.

- If a current 'leaves' the dotted terminal of a coil, the reference polarity of the voltage induced in the other coil is negative at its dotted terminal.

Explanation:

In the time domain, the self-induced and mutually induced voltage depend on the rate of change of current. For example, the voltage induced in winding L_2 due to the current I_1 is:

$$v_{L2}(t) = M\frac{dI_1}{dt} \qquad \text{Volts}$$

In the frequency domain, this becomes:

$$v_{L2}(\omega) = j\omega M I_1 \qquad \text{Volts}$$

Here, 'j' is the complex operator $j = \sqrt{-1}$.

Similarly, the voltage induced into coil L_1 by the current I_1 is:

$$v_{L1}(t) = L_1\frac{dI_1}{dt} \qquad \text{Volts}$$

In the frequency domain:

$$v_{L1}(\omega) = j\omega L_1 I_1 \qquad \text{Volts}$$

The text book definition for mutual inductance 'M' is as follows: $M = k\sqrt{L_1 L_2}$ where k is a number in the range 0 to 1. A value of k = 0 implies no coupling whereas k = 1 suggests very tight coupling. For example, tightly twisted wires wound on a closed ferrite high permeability core will probably have a k value higher than 0.99.

4 Tandem Bridge Analysis

Figure 10 shows the basic arrangement of the Tandem Bridge connected with 50Ω loads on the forward and reverse ports. Complexity occurs because transformers generate voltages in response to current changes. The method used to analyse the bridge is to add up the port voltages and systematically eliminate each of the currents. This results in a lot of tedious complexity. The following diagram shows a possible guess at currents at a given time instance and the associated induced voltages.

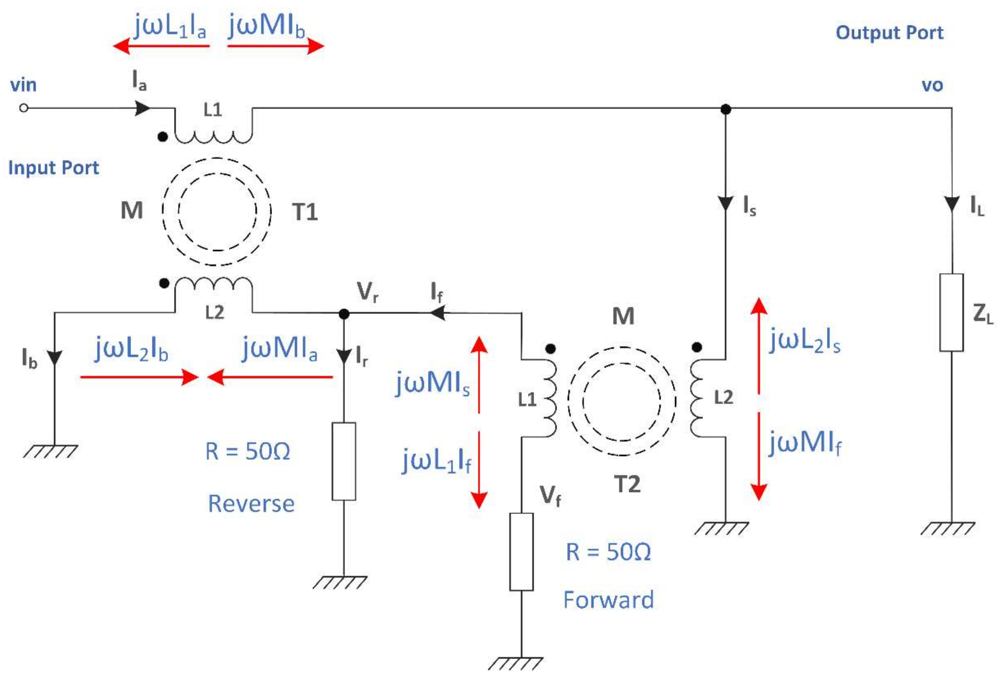


Figure 10 - Tandem Bridge Complete with Induced Voltages

Summing up the voltages across the forward power port resistor:

$$V_f = j\omega L_2 I_b - j\omega M I_a - j\omega M I_s + j\omega L_1 I_f \qquad \text{Equation 1}$$

And, at the reflected power port:

$$V_r = j\omega L_2 I_b - j\omega M I_a \qquad \text{Equation 2}$$

So, substituting equation 2 into equation 1

$$V_f = V_r - j\omega M I_s + j\omega L_1 I_f \qquad \text{Equation 3}$$

By ohms law

$$I_f = -\frac{V_f}{R} \qquad\qquad I_r = \frac{V_r}{R}$$

So

$$I_b = I_f - I_r = \frac{-\left(V_f + V_r\right)}{R}$$ Equation 4

Looking at the output port

$$V_o = j\omega L_2 I_s - j\omega M\, I_f$$ Equation 5

So $$I_s = \frac{V_o + j\omega M\, I_f}{j\omega L_2} = \frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2}$$ Equation 6

Also, at the output port:

$$I_a = I_s + I_L \quad \text{and} \quad I_L = \frac{V_o}{Z}$$ Equation 7

So

$$I_a = \frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}$$ Equation 8

Now the preliminaries are complete, find the forward voltage term V_r by substituting equation 4 and 8 into equation 2

$$V_r = j\omega L_2 I_b - j\omega M I_a$$ Equation 2

$$V_r = j\omega L_2\left(\frac{-V_f - V_r}{R}\right) - j\omega M\left(\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}\right)$$

Multiply out and separate terms

$$V_r\left(1 + \frac{j\omega L_2}{R}\right) = V_f\left(\frac{j\omega M^2}{RL_2} - \frac{j\omega L_2}{R}\right) - \frac{j\omega MV_o}{Z} - \frac{MV_o}{L_2}$$ Equation 9

So

$$V_r = \frac{V_f\left(\frac{j\omega M^2}{RL_2} - \frac{j\omega L_2}{R}\right) - \frac{j\omega MV_o}{Z} - \frac{MV_o}{L_2}}{1 + \frac{j\omega L_2}{R}}$$ Equation 10 ****

Equation 10 will be needed later. Now substitute equation 6 and $I_f = -\frac{V_f}{R}$ into equation 3

$$V_f = V_r - j\omega M I_s + j\omega L_1 I_f \qquad \text{Equation 3}$$

$$V_f = V_r - j\omega M\left(\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2}\right) + j\omega L_1\left(-\frac{V_f}{R}\right)$$

Collect terms and tidy up

$$V_f\left(1 - j\omega\frac{M^2}{RL_2} + j\omega\frac{L_1}{R}\right) = V_r - \frac{MV_o}{L_2} \qquad \text{Equation 11}$$

Or

$$V_f = \frac{V_r - \frac{MV_o}{L_2}}{1 - j\omega\frac{M^2}{RL_2} + j\omega\frac{L_1}{R}}$$

Now substitute $M = k\sqrt{L_1 L_2}$

$$V_f = \frac{V_r - kV_o\sqrt{\frac{L_1}{L_2}}}{1 - j\omega\frac{k^2 L_1}{R} + j\omega\frac{L_1}{R}}$$

So

$$V_f = \frac{V_r - kV_o\sqrt{\frac{L_1}{L_2}}}{1 + j\omega\frac{L_1}{R}\left(1 - k^2\right)} \qquad \text{Equation 12***}$$

It is apparent that when V_r is small and k=1 then $V_f \approx -V_o\sqrt{\frac{L_1}{L_2}}$. That is an antiphase version of the output signal reduced by a factor determined by the transformer turns ratio. Equation 9 and equation 11 are simultaneous solutions. For a given V_o, V_f or V_r can be calculated.

Get the reflected voltage V_r :

Substitute equation 12 into equation 9

$$V_r\left(1+\frac{j\omega L_2}{R}\right)=V_f\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)-\frac{j\omega MV_o}{Z}-\frac{MV_o}{L_2}$$ Equation 9

$$V_r\left(1+\frac{j\omega L_2}{R}\right)=\left(\frac{V_r-\frac{MV_o}{L_2}}{1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}}\right)\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)-\frac{j\omega MV_o}{Z}-\frac{MV_o}{L_2}$$

Multiply both sides by the first denominator: $\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}\right)$

$$V_r\left(1+\frac{j\omega L_2}{R}\right)\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}\right)=\left(V_r-\frac{MV_o}{L_2}\right)\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)-\left(\frac{j\omega MV_o}{Z}+\frac{MV_o}{L_2}\right)\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}\right)$$

To keep this on the page, the LHS of the above equation is calculated first:

$$LHS=V_r\left(1+\frac{j\omega L_2}{R}\right)\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}\right)=V_r\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}+j\omega\frac{L_2}{R}+\frac{\omega^2M^2}{R^2}-\frac{\omega^2L_2L_1}{R^2}\right)$$

Two terms from the RHS $V_r\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)$ can be transferred to the LHS.

$$LHS=V_r\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}+j\omega\frac{L_2}{R}+\frac{\omega^2M^2}{R^2}-\frac{\omega^2L_2L_1}{R^2}\right)-V_r\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)$$

So, after collecting terms

$$LHS=V_r\left(1-j\omega\frac{2M^2}{RL_2}+j\omega\frac{L_1}{R}+j\omega\frac{2L_2}{R}+\frac{\omega^2M^2}{R^2}-\frac{\omega^2L_2L_1}{R^2}\right)$$

Substituting $M=k\sqrt{L_1L_2}$

$$LHS=V_r\left(1-j\omega\frac{2k^2L_1L_2}{RL_2}+j\omega\frac{L_1}{R}+j\omega\frac{2L_2}{R}+\frac{\omega^2k^2L_1L_2}{R^2}-\frac{\omega^2L_1L_2}{R^2}\right)$$

Tidying up

$$LHS = V_r\left(1 + j\omega\frac{2L_2}{R} + j\omega\frac{L_1}{R}\left(1-2k^2\right) + \frac{\omega^2 L_1 L_2 (k^2-1)}{R^2}\right)$$

The RHS is (Remember two terms were transferred to the LHS a few lines up):

$$RHS = -\frac{MV_o}{L_2}\left(\frac{j\omega M^2}{RL_2} - \frac{j\omega L_2}{R}\right) - \left(\frac{j\omega MV_o}{Z} + \frac{MV_o}{L_2}\right)\left(1 - j\omega\frac{M^2}{RL_2} + j\omega\frac{L_1}{R}\right)$$

Multiplying out:

$$RHS = -j\omega\frac{M^3V_o}{RL_2L_2} + j\omega\frac{MV_o}{R} - \left(\frac{j\omega MV_o}{Z} + \omega^2\frac{M^3V_o}{ZRL_2} - \frac{\omega^2 ML_1V_o}{ZR} + \frac{MV_o}{L_2} - j\omega\frac{M^3V_o}{RL_2L_2} + j\omega\frac{ML_1V_o}{RL_2}\right)$$

$$RHS = -j\omega\frac{M^3V_o}{RL_2L_2} + j\omega\frac{MV_o}{R} - \frac{j\omega MV_o}{Z} - \omega^2\frac{M^3V_o}{ZRL_2} + \frac{\omega^2 ML_1V_o}{ZR} - \frac{MV_o}{L_2} + j\omega\frac{M^3V_o}{RL_2L_2} - j\omega\frac{ML_1V_o}{RL_2}$$

Tidy up – one miserable term cancels!

$$RHS = j\omega\frac{MV_o}{R} - j\omega\frac{MV_o}{Z} - \omega^2\frac{M^3V_o}{ZRL_2} + \omega^2\frac{ML_1V_o}{ZR} - \frac{MV_o}{L_2} - j\omega\frac{ML_1V_o}{RL_2}$$

Substituting $M = k\sqrt{L_1L_2}$

$$RHS = j\omega\frac{k\sqrt{L_1L_2}V_o}{R} - j\omega\frac{k\sqrt{L_1L_2}V_o}{Z} - \omega^2\frac{k^3L_1L_2\sqrt{L_1L_2}V_o}{ZRL_2} + \omega^2\frac{k\sqrt{L_1L_2}L_1V_o}{ZR} - \frac{k\sqrt{L_1L_2}V_o}{L_2} - j\omega\frac{k\sqrt{L_1L_2}L_1V_o}{RL_2}$$

$$RHS = j\omega k\sqrt{L_1L_2}V_o\left(\frac{1}{R} - \frac{1}{Z}\right) + \omega^2\frac{kL_1\sqrt{L_1L_2}V_o}{ZR}\left(1-k^2\right) - kV_o\sqrt{\frac{L_1}{L_2}} - j\omega\frac{kL_1V_0}{R}\sqrt{\frac{L_1}{L_2}}$$

Now RHS and LHS are complete, the final expression for V_f is given by

$$V_r = \frac{j\omega k\sqrt{L_1L_2}V_o\left(\frac{1}{R} - \frac{1}{Z}\right) + \omega^2\frac{kL_1\sqrt{L_1L_2}V_o}{ZR}\left(1-k^2\right) - kV_o\sqrt{\frac{L_1}{L_2}} - j\omega\frac{kL_1V_0}{R}\sqrt{\frac{L_1}{L_2}}}{1 + j\omega\frac{2L_2}{R} + j\omega\frac{L_1}{R}\left(1-2k^2\right) + \frac{\omega^2 L_1L_2(k^2-1)}{R^2}}$$

Finally, multiply top and bottom by R. This could be tidied up further. However, it is too complex to calculate by hand and V_r will never be beautiful so I fail to see the advantage.

The Final equations:

$$V_r = \frac{V_0\left(j\omega k\sqrt{L_1L_2}\left(1-\frac{R}{Z}\right)+\omega^2\frac{kL_1\sqrt{L_1L_2}}{Z}\left(1-k^2\right)-kR\sqrt{\frac{L_1}{L_2}}-j\omega kL_1\sqrt{\frac{L_1}{L_2}}\right)}{R+j2\omega L_2+j\omega L_1\left(1-2k^2\right)+\frac{\omega^2L_1L_2}{R}(k^2-1)}$$

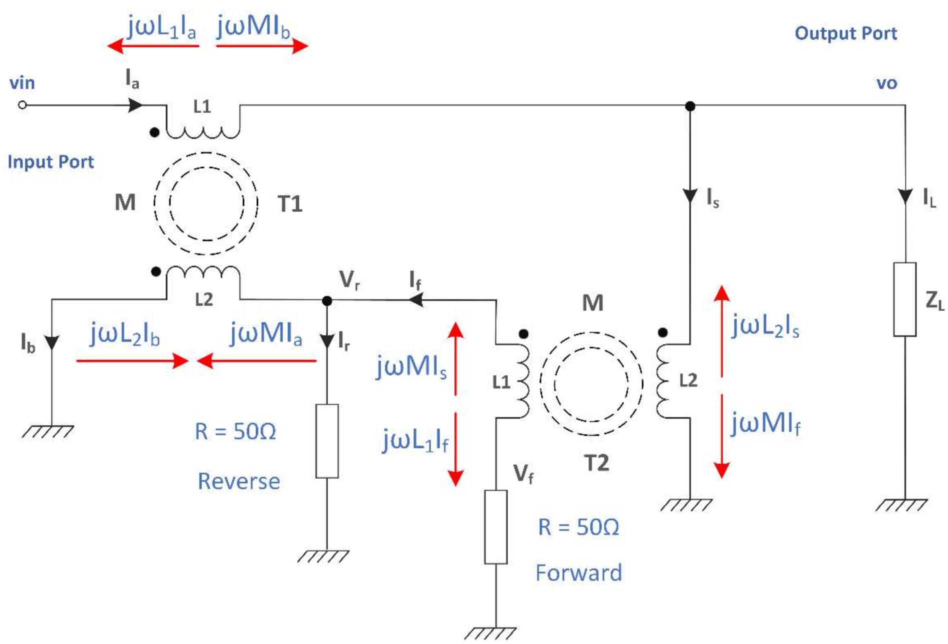
Equation 13***

$$V_f = \frac{V_r - kV_o\sqrt{\frac{L_1}{L_2}}}{1+j\omega\frac{L_1}{R}\left(1-k^2\right)}$$

Equation 12***

4.1 Input Impedance and Insertion Loss

Repeating the Tandem bridge voltage diagram:



Input Impedance

The input voltage vin is:

$$V_{in} = V_0 + j\omega L_1 I_a - j\omega M I_b$$

The input impedance $Z_{in} = \frac{V_{in}}{I_a}$

So

$$Z_{in} = \frac{V_{in}}{I_a} = \frac{V_0 - j\omega M I_b}{I_a} + j\omega L_1$$

Substituting for

$$I_a = \frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}$$

Equation 8 - See earlier in this document

And

$$I_b = I_f - I_r = \frac{-(V_f + V_r)}{R}$$

Equation 4 - See earlier in this document

Hence

$$Z_{in} = \frac{V_0 + j\omega M\left(\frac{V_f + V_r}{R}\right)}{\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}} + j\omega L_1$$

Substituting $M = k\sqrt{L_1 L_2}$

$$Z_{in} = \frac{V_0 + j\omega k\sqrt{L_1 L_2}\left(\frac{V_f + V_r}{R}\right)}{\frac{V_o}{j\omega L_2} - \frac{k\sqrt{L_1 L_2}V_f}{RL_2} + \frac{V_o}{Z}} + j\omega L_1$$

Equation 14

Insertion Loss:

Start with the input voltage:

$$V_{in} = V_0 + j\omega L_1 I_a - j\omega M I_b$$

Substitute Equations 8 and 4:

$$V_{in} = V_0 + j\omega L_1\left(\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}\right) + j\omega M\left(\frac{V_f + V_r}{R}\right)$$

The insertion loss in dB is

$$I_L = 20.\log\left(\left|\frac{V_0}{V_{in}}\right|\right)$$

or

$$I_L = 20.\log\left(\left|\frac{V_0}{V_0 + j\omega L_1\left(\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}\right) + j\omega M\left(\frac{V_f + V_r}{R}\right)}\right|\right)$$

Finally, substitute $M = k\sqrt{L_1L_2}$

$$I_L = 20.\log\left(\left|\frac{V_0}{V_0 + j\omega L_1\left(\frac{V_o}{j\omega L_2} - \frac{k\sqrt{L_1L_2}V_f}{RL_2} + \frac{V_o}{Z}\right) + j\omega k\sqrt{L_1L_2}\left(\frac{V_f + V_r}{R}\right)}\right|\right)$$ Equation 15

End of document - phew