1 Appendix A - Outline of a Proof for Equation 12, 13, 14 and 15 - Preliminaries

Mutual inductance is necessary to analyse coupled circuits. To be successful, it is essential to follow a set of rules and remain consistent in applying signs throughout. It is necessary to guess the initial current directions and then calculate associated voltages accepting that some guesses may be incorrect. If the analysis is consistent and applied robotically, the analysis works out correctly in the wash. The voltages induce into windings come in two forms:

(1) self-induced voltages (you could call these back-emfs if you have too); and

(2) mutually induced voltage that are generated by changing currents in the other winding.

The relative polarities of the voltages are described by the dot convention as shown in Figure 1. Of course, all voltages and currents are assumed RF sinusoids.

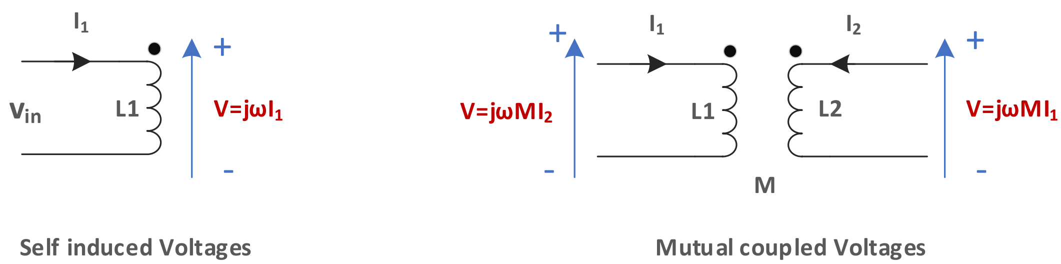


Figure 1 - Application of the Dot Convention

Transformer Dot Convention for mutual inductance is described in words as:

- If a current 'enters' into the dotted terminal of a coil, then the polarity of the induced voltage in the other coil is positive at its dotted terminal.

- If a current 'leaves' the dotted terminal of a coil, the reference polarity of the voltage induced in the other coil is negative at its dotted terminal.

Explanation:

In the time domain, the self-induced and mutually induced voltage depend on the rate of change of current. For example, the voltage induced in winding L_2 due to the current I_1 is:

$$v_{L2}(t) = M\frac{dI_1}{dt} \qquad \text{Volts}$$

In the frequency domain, this becomes:

$$v_{L2}(\omega) = j\omega M I_1 \qquad \text{Volts}$$

Here, 'j' is the complex operator $j = \sqrt{-1}$.

Similarly, the voltage induced into coil L_1 by the current I_1 is:

$$v_{L1}(t) = L_1\frac{dI_1}{dt} \qquad \text{Volts}$$

In the frequency domain:

$$v_{L1}(\omega) = j\omega L_1 I_1 \qquad \text{Volts}$$

The text book definition for mutual inductance 'M' is as follows: $M = k\sqrt{L_1L_2}$ where k is a number in the range 0 to 1. A value of k = 0 implies no coupling whereas k = 1 suggests very tight coupling. For example, tightly twisted wires wound on a closed ferrite high permeability core will probably have a k value higher than 0.99.

2 Tandem Bridge Analysis

Figure 2 shows the basic arrangement of the Tandem Bridge connected with 50Ω loads on the forward and reverse ports. Complexity occurs because transformers generate voltages in response to current changes. The method used to analyse the bridge is to add up the port voltages and systematically eliminate each of the currents. This results in a lot of tedious complexity. The following diagram shows a possible guess at currents at a given time instance and the associated induced voltages.

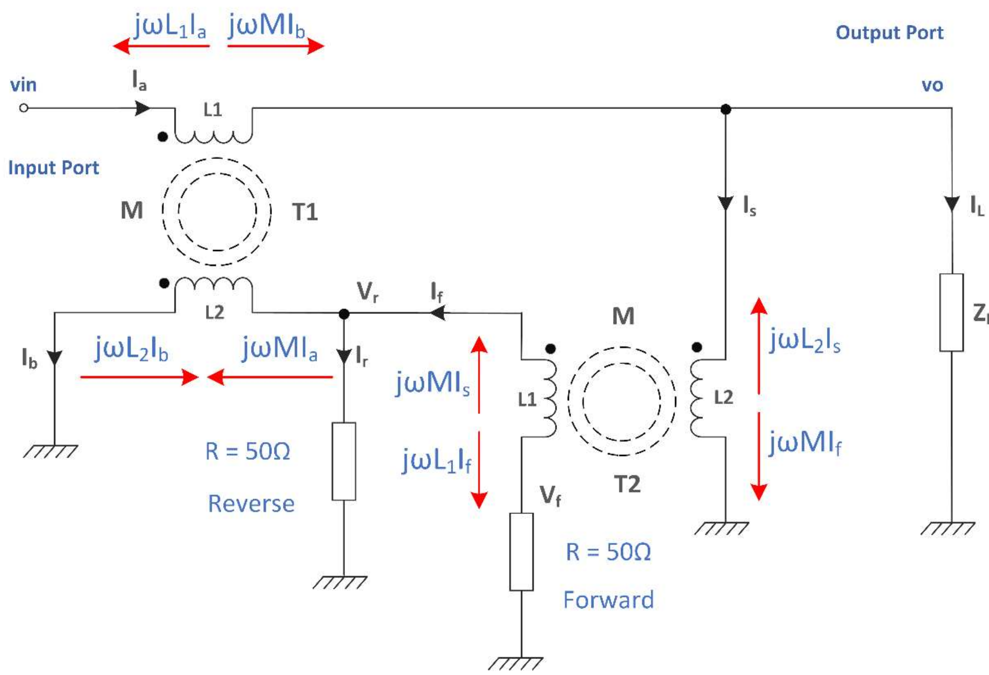


Figure 2 - Tandem Bridge Complete with Induced Voltages

Summing up the voltages across the forward power port resistor:

$$V_f = j\omega L_2 I_b - j\omega M I_a - j\omega M I_s + j\omega L_1 I_f \qquad \text{Equation 1}$$

And, at the reflected power port:

$$V_r = j\omega L_2 I_b - j\omega M I_a \qquad \text{Equation 2}$$

So, substituting equation 2 into equation 1

$$V_f = V_r - j\omega M I_s + j\omega L_1 I_f \qquad \text{Equation 3}$$

By ohms law

$$I_f = -\frac{V_f}{R} \qquad I_r = \frac{V_r}{R}$$

So

$$I_b = I_f - I_r = \frac{-\left(V_f + V_r\right)}{R}$$ Equation 4

Looking at the output port

$$V_o = j\omega L_2 I_s - j\omega M\, I_f$$ Equation 5

So $$I_s = \frac{V_o + j\omega\, M\, I_f}{j\omega\, L_2} = \frac{V_o}{j\omega\, L_2} - \frac{MV_f}{RL_2}$$ Equation 6

Also, at the output port:

$I_a = I_s + I_L$ and $I_L = \dfrac{V_o}{Z}$ Equation 7

So

$$I_a = \frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}$$ Equation 8

Now the preliminaries are complete, find the forward voltage term V_r by substituting equation 4 and 8 into equation 2

$$V_r = j\omega\, L_2 I_b - j\omega\, MI_a$$ Equation 2

$$V_r = j\omega L_2\left(\frac{-V_f - V_r}{R}\right) - j\omega M\left(\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}\right)$$

Multiply out and separate terms

$$V_r\left(1 + \frac{j\omega L_2}{R}\right) = V_f\left(\frac{j\omega M^2}{RL_2} - \frac{j\omega L_2}{R}\right) - \frac{j\omega MV_o}{Z} - \frac{MV_o}{L_2}$$ Equation 9

So

$$V_r = \frac{V_f\left(\frac{j\omega M^2}{RL_2} - \frac{j\omega L_2}{R}\right) - \frac{j\omega MV_o}{Z} - \frac{MV_o}{L_2}}{1 + \frac{j\omega L_2}{R}}$$ Equation 10 ****

Equation 10 will be needed later. Now substitute equation 6 and $I_f = -\frac{V_f}{R}$ into equation 3

$$V_f = V_r - j\omega M I_s + j\omega L_1 I_f$$ Equation 3

$$V_f = V_r - j\omega M\left(\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2}\right) + j\omega L_1\left(-\frac{V_f}{R}\right)$$

Collect terms and tidy up

$$V_f\left(1 - j\omega\frac{M^2}{RL_2} + j\omega\frac{L_1}{R}\right) = V_r - \frac{MV_o}{L_2}$$ Equation 11

Or

$$V_f = \frac{V_r - \frac{MV_o}{L_2}}{1 - j\omega\frac{M^2}{RL_2} + j\omega\frac{L_1}{R}}$$

Now substitute $M = k\sqrt{L_1 L_2}$

$$V_f = \frac{V_r - kV_o\sqrt{\frac{L_1}{L_2}}}{1 - j\omega\frac{k^2 L_1}{R} + j\omega\frac{L_1}{R}}$$

So

$$V_f = \frac{V_r - kV_o\sqrt{\frac{L_1}{L_2}}}{1 + j\omega\frac{L_1}{R}\left(1 - k^2\right)}$$ Equation 12***

It is apparent that when V_r is small and k=1 then $V_f \approx -V_o\sqrt{\frac{L_1}{L_2}}$. That is an antiphase version of the output signal reduced by a factor determined by the transformer turns ratio. Equation 9 and equation 11 are simultaneous solutions. For a given V_o, V_f or V_r can be calculated.

Get the reflected voltage V_r :

Substitute equation 12 into equation 9

$$V_r\left(1+\frac{j\omega L_2}{R}\right)=V_f\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)-\frac{j\omega MV_o}{Z}-\frac{MV_o}{L_2}$$ Equation 9

$$V_r\left(1+\frac{j\omega L_2}{R}\right)=\left(\frac{V_r-\frac{MV_o}{L_2}}{1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}}\right)\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)-\frac{j\omega MV_o}{Z}-\frac{MV_o}{L_2}$$

Multiply both sides by the first denominator: $\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}\right)$

$$V_r\left(1+\frac{j\omega L_2}{R}\right)\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}\right)=\left(V_r-\frac{MV_o}{L_2}\right)\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)-\left(\frac{j\omega MV_o}{Z}+\frac{MV_o}{L_2}\right)\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}\right)$$

To keep this on the page, the LHS of the above equation is calculated first:

$$LHS=V_r\left(1+\frac{j\omega L_2}{R}\right)\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}\right)=V_r\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}+j\omega\frac{L_2}{R}+\frac{\omega^2M^2}{R^2}-\frac{\omega^2L_2L_1}{R^2}\right)$$

Two terms from the RHS $V_r\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)$ can be transferred to the LHS.

$$LHS=V_r\left(1-j\omega\frac{M^2}{RL_2}+j\omega\frac{L_1}{R}+j\omega\frac{L_2}{R}+\frac{\omega^2M^2}{R^2}-\frac{\omega^2L_2L_1}{R^2}\right)-V_r\left(\frac{j\omega M^2}{RL_2}-\frac{j\omega L_2}{R}\right)$$

So, after collecting terms

$$LHS=V_r\left(1-j\omega\frac{2M^2}{RL_2}+j\omega\frac{L_1}{R}+j\omega\frac{2L_2}{R}+\frac{\omega^2M^2}{R^2}-\frac{\omega^2L_2L_1}{R^2}\right)$$

Substituting $M=k\sqrt{L_1L_2}$

$$LHS=V_r\left(1-j\omega\frac{2k^2L_1L_2}{RL_2}+j\omega\frac{L_1}{R}+j\omega\frac{2L_2}{R}+\frac{\omega^2k^2L_1L_2}{R^2}-\frac{\omega^2L_1L_2}{R^2}\right)$$

Tidying up

$$LHS = V_r\left(1 + j\omega\frac{2L_2}{R} + j\omega\frac{L_1}{R}\left(1-2k^2\right) + \frac{\omega^2 L_1 L_2 (k^2-1)}{R^2}\right)$$

The RHS is (Remember two terms were transferred to the LHS a few lines up):

$$RHS = -\frac{MV_o}{L_2}\left(\frac{j\omega M^2}{RL_2} - \frac{j\omega L_2}{R}\right) - \left(\frac{j\omega MV_o}{Z} + \frac{MV_o}{L_2}\right)\left(1 - j\omega\frac{M^2}{RL_2} + j\omega\frac{L_1}{R}\right)$$

Multiplying out:

$$RHS = -j\omega\frac{M^3V_o}{RL_2L_2} + j\omega\frac{MV_o}{R} - \left(\frac{j\omega MV_o}{Z} + \omega^2\frac{M^3V_o}{ZRL_2} - \frac{\omega^2 ML_1V_o}{ZR} + \frac{MV_o}{L_2} - j\omega\frac{M^3V_o}{RL_2L_2} + j\omega\frac{ML_1V_o}{RL_2}\right)$$

$$RHS = -j\omega\frac{M^3V_o}{RL_2L_2} + j\omega\frac{MV_o}{R} - \frac{j\omega MV_o}{Z} - \omega^2\frac{M^3V_o}{ZRL_2} + \frac{\omega^2 ML_1V_o}{ZR} - \frac{MV_o}{L_2} + j\omega\frac{M^3V_o}{RL_2L_2} - j\omega\frac{ML_1V_o}{RL_2}$$

Tidy up – one miserable term cancels!

$$RHS = j\omega\frac{MV_o}{R} - j\omega\frac{MV_o}{Z} - \omega^2\frac{M^3V_o}{ZRL_2} + \omega^2\frac{ML_1V_o}{ZR} - \frac{MV_o}{L_2} - j\omega\frac{ML_1V_o}{RL_2}$$

Substituting $M = k\sqrt{L_1L_2}$

$$RHS = j\omega\frac{k\sqrt{L_1L_2}V_o}{R} - j\omega\frac{k\sqrt{L_1L_2}V_o}{Z} - \omega^2\frac{k^3L_1L_2\sqrt{L_1L_2}V_o}{ZRL_2} + \omega^2\frac{k\sqrt{L_1L_2}L_1V_o}{ZR} - \frac{k\sqrt{L_1L_2}V_o}{L_2} - j\omega\frac{k\sqrt{L_1L_2}L_1V_o}{RL_2}$$

$$RHS = j\omega k\sqrt{L_1L_2}V_o\left(\frac{1}{R} - \frac{1}{Z}\right) + \omega^2\frac{kL_1\sqrt{L_1L_2}V_o}{ZR}\left(1-k^2\right) - kV_o\sqrt{\frac{L_1}{L_2}} - j\omega\frac{kL_1V_0}{R}\sqrt{\frac{L_1}{L_2}}$$

Now RHS and LHS are complete, the final expression for V_f is given by

$$V_r = \frac{j\omega k\sqrt{L_1L_2}V_o\left(\frac{1}{R} - \frac{1}{Z}\right) + \omega^2\frac{kL_1\sqrt{L_1L_2}V_o}{ZR}\left(1-k^2\right) - kV_o\sqrt{\frac{L_1}{L_2}} - j\omega\frac{kL_1V_0}{R}\sqrt{\frac{L_1}{L_2}}}{1 + j\omega\frac{2L_2}{R} + j\omega\frac{L_1}{R}\left(1-2k^2\right) + \frac{\omega^2 L_1L_2(k^2-1)}{R^2}}$$

Finally, multiply top and bottom by R. This could be tidied up further. However, it is too complex to calculate by hand and V_r will never be beautiful so I fail to see the advantage.

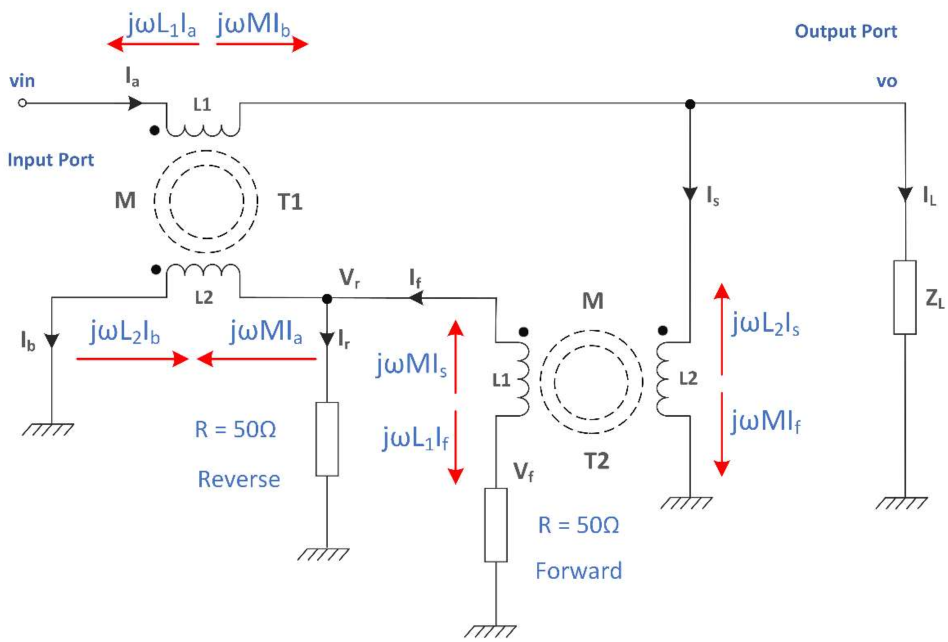
The Final equations:

$$V_r = \frac{V_0\left(j\omega k\sqrt{L_1L_2}\left(1-\frac{R}{Z}\right) + \omega^2 \frac{kL_1\sqrt{L_1L_2}}{Z}\left(1-k^2\right) - kR\sqrt{\frac{L_1}{L_2}} - j\omega kL_1\sqrt{\frac{L_1}{L_2}}\right)}{R + j2\omega L_2 + j\omega L_1\left(1-2k^2\right) + \frac{\omega^2 L_1L_2}{R}(k^2-1)}$$ Equation 13***

$$V_f = \frac{V_r - kV_o\sqrt{\frac{L_1}{L_2}}}{1 + j\omega\frac{L_1}{R}\left(1-k^2\right)}$$ Equation 12***

2.1 Input Impedance and Insertion Loss

Repeating the Tandem bridge voltage diagram:



Input Impedance

The input voltage vin is:

$$V_{in} = V_0 + j\omega L_1 I_a - j\omega M I_b$$

The input impedance $Z_{in} = \frac{V_{in}}{I_a}$

So

$$Z_{in} = \frac{V_{in}}{I_a} = \frac{V_0 - j\omega M I_b}{I_a} + j\omega L_1$$

Substituting for

$$I_a = \frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}$$

Equation 8 - See earlier in this document

And

$$I_b = I_f - I_r = \frac{-\left(V_f + V_r\right)}{R}$$

Equation 4 - See earlier in this document

Hence

$$Z_{in} = \frac{V_0 + j\omega M\left(\frac{V_f + V_r}{R}\right)}{\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}} + j\omega L_1$$

Substituting $M = k\sqrt{L_1 L_2}$

$$Z_{in} = \frac{V_0 + j\omega k\sqrt{L_1 L_2}\left(\frac{V_f + V_r}{R}\right)}{\frac{V_o}{j\omega L_2} - \frac{k\sqrt{L_1 L_2}V_f}{RL_2} + \frac{V_o}{Z}} + j\omega L_1$$

Equation 14

Insertion Loss:

Start with the input voltage:

$$V_{in} = V_0 + j\omega L_1 I_a - j\omega M I_b$$

Substitute Equations 8 and 4:

$$V_{in} = V_0 + j\omega L_1\left(\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}\right) + j\omega M\left(\frac{V_f + V_r}{R}\right)$$

The insertion loss in dB is

$$I_L = 20.\log\left(\left|\frac{V_0}{V_{in}}\right|\right)$$

or

$$I_L = 20.\log\left(\left|\frac{V_0}{V_0 + j\omega L_1\left(\frac{V_o}{j\omega L_2} - \frac{MV_f}{RL_2} + \frac{V_o}{Z}\right) + j\omega M\left(\frac{V_f + V_r}{R}\right)}\right|\right)$$

Finally, substitute $M = k\sqrt{L_1 L_2}$

$$I_L = 20.\log\left(\left|\frac{V_0}{V_0 + j\omega L_1\left(\frac{V_o}{j\omega L_2} - \frac{k\sqrt{L_1 L_2}V_f}{RL_2} + \frac{V_o}{Z}\right) + j\omega k\sqrt{L_1 L_2}\left(\frac{V_f + V_r}{R}\right)}\right|\right) \qquad \text{Equation 15}$$

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